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Promotion, Evacuation and Cactus groups

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Abstract. The promotion operator on rectangular standard tableaux can be generalised to an operator acting on the invariant highest weight words in the tensor power of a crystal. For the vector representation of a symplectic group the Sundaram correspondence is an injective map to perfect matchings. We show that this map intertwines promotion and rotation. For the adjoint representation of a general linear group we construct a similar map to permutations. We show that this map also intertwines promotion and rotation. These results are proved using an approach to the action of the cactus group using a generalisation of local rules and growth diagrams.

Keywords: promotion, evacuation, cactus group

1 Introduction

Our journey begins with the discovery that Sundaram's map from oscillating tableaux to perfect matchings, regarded as chord diagrams (as in Figure 1), intertwines promotion and rotation, see Theorem 2.4. Oscillating tableaux are highest weight words in a tensor power of the crystal of the vector representation of the symplectic group $\mathrm{Sp}(2n)$, and promotion is a natural generalisation of Schützenberger's promotion map on standard Young tableaux.

Similarly, there is a natural map from Stembridge's alternating tableaux, the highest weight words in the r-th tensor power of the crystal for the adjoint representation of the general linear group $\mathrm{GL}(n)$, to permutations. It turns out that this map intertwines promotion and rotation provided that $n \geq r$, see Theorem 2.8.

A convenient setting for these variants of promotion are the cactus groups, introduced by Devadoss [2, def 6.1.2] and placed into our context by Henriques and Kamnitzer [3]. These are infinite groups related to coboundary categories in a similar way as the braid groups are related to braided categories. Thus, our goal is to make the effect of the cactus groups in the coboundary category of crystals of a complex reductive Lie algebra $\mathfrak{g}$ transparent. To do so, we use the local rules discovered by van Leeuwen [4],

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generalising the classical local rule for jeu de taquin by Fomin to all minuscule representations of Lie groups. The relation between these local rules and the action of the cactus groups was established by Lenart [5].

Let $C = C_1 \otimes \cdots \otimes C_r$ be an r-fold tensor product of crystals. Then the generators $\mathrm{s}_{p,q}$, for $1 \leq p < q \leq r$, of the cactus group map highest weight words of C bijectively to highest weight words of $C_1 \otimes \cdots \otimes C_{p-1} \otimes C_q \otimes \cdots \otimes C_p \otimes C_{q+1} \otimes \cdots \otimes C_r$.

For example, when $\mathfrak{g}$ is the Lie algebra of the special linear group $\mathrm{SL}(n)$, and $C_1 = \cdots = C_r$ is the crystal of its vector representation, the highest weight words of C are standard Young tableaux of size r with at most n columns. Then, the generator $\mathrm{s}_{1,r}$ of the cactus group is precisely Schützenberger's evacuation, and $\mathrm{s}_{1,r}\,\mathrm{s}_{2,r}$ is Schützenberger's promotion. As an aside, we remark that in this case the generators $\mathrm{s}_{i,i+2}$ encode Assaf's dual equivalence graph.

More generally, when C_i is the crystal of the μ_i-th exterior power of the vector representation then the highest weight words of C are semistandard Young tableaux of weight μ, with at most n columns: the weight of the i-th letter specifies the columns in which the number i appears. Again, $\mathrm{s}_{1,r}$ acts as evacuation and $\mathrm{s}_{1,r}\,\mathrm{s}_{2,r}$ as promotion. Using evacuation as a building block, the action of the cactus groups on semistandard Young tableaux was studied by Chmutov, Glick and Pylyavskyy [1].

By analogy we call $\mathrm{ev}\, w = \mathrm{s}_{1,r}\, w$ the evacuation of a highest weight word w, and $\mathrm{pr}\, w = \mathrm{s}_{1,r}\,\mathrm{s}_{2,r}\, w$ its promotion. The promotion of w can be obtained as follows:

let w' be w without its first letter,

let w'' be the unique highest weight word in the same component as w',

append the uniquely determined letter to w'', such that the result has the same weight as w.

This general definition of the promotion operator has been studied in connection with the cyclic sieving phenomenon. Rhoades established a cyclic sieving phenomenon for the promotion operator acting on rectangular standard tableaux. This was generalised to promotion on invariant words in the crystal of a minuscule representation by Fontaine and Kamnitzer, using the geometric Satake correspondence. A cyclic sieving phenomenon for the promotion operator acting on invariant words in any crystal was given by Westbury [14], exploiting the fact that Lusztig's canonical basis for invariant tensors is preserved by promotion.

Cyclic sieving phenomena for perfect matchings and permutations together with rotation were established by Rubey and Westbury [8, 9]. There, a basis of the space of invariant tensors which is invariant under rotation was constructed, in an elementary way. The construction involved the two fundamental theorems of invariant theory, asserting that the tensors of chord diagrams are a basis of the space of invariant tensors.

The promotion and rotation operators on invariant tensors were shown to agree by Westbury [14], in the following sense. It is true in general that, given a vector space

Figure 1: A 3-noncrossing perfect matching and a permutation as chord diagrams.

with a linear operator of finite order and two bases each preserved by the operator then there is a bijection between the two bases which intertwines the two actions of the operator. In particular, this implies for our setting that there exists a bijection between chord diagrams and invariant words which intertwines rotation and promotion. However constructing such a bijection explicitly remained an open problem, which we solve here.

2 Results

We aim at making the action of the cactus group on the highest weight words of a tensor power of certain representations transparent. Our approach works best for tensor products of minuscule representations of a Lie group. A representation is *minuscule* if the Weyl group W of the Lie group acts transitively on the weights of the representation: the set of weights forms a single orbit under the action of W. The non-trivial minuscule representations are:

Type A_n: All exterior powers of the vector representation.

Type B_n: The spin representation.

Type C_n: The vector representation.

Type D_n: The vector representation and the two half-spin representations.

Type E_6: The two fundamental representations of dimension 27.

Type E_7: The fundamental representation of dimension 56.

There are no nontrivial minuscule representations in types G_2, F_4 or E_8.

For tensor products of exterior powers of the vector representation of $\mathrm{GL}(n)$, the action of the cactus group is known, as already mentioned in the introduction.

Highest weight words of weight zero of $\otimes^r S$, where S is the the spin representation of the spin group $\mathrm{Spin}(2n+1)$ can be identified directly with fans of n Dyck paths of length r. One can show that ev acts on these as reversal.

The vector representation of the odd orthogonal group $\mathrm{SO}(2n+1)$ is not minuscule, but appears as a direct summand in $S \otimes S$. In particular, highest weight words of weight zero of a tensor power of the vector representation of $\mathrm{SO}(3)$ can be identified with noncrossing set partitions without singletons, and pr acts on these as rotation.

However, our main contributions concern the vector representation of $\mathrm{Sp}(2n)$ and the adjoint representation of $\mathrm{GL}(n)$ - regarded as the tensor product of the vector representation and its dual.

2.1 The vector representation of the symplectic groups

Definition 2.1 (Sundaram [13])**.** *An n-*symplectic oscillating tableau *of length r and (final) shape μ is a sequences of partitions*

$$\emptyset = \mu^0, \mu^1, \dots, \mu^r = \mu$$

such that the Ferrers diagrams of two consecutive partitions differ by exactly one box, and each partition μ^i has at most n non-zero parts.

Proposition 2.2. *Let C be the crystal corresponding to $\otimes^r V$, where V is the vector representation of the symplectic group $\mathrm{Sp}(2n)$. Then the highest weight words of C are obtained from n-symplectic oscillating tableaux by considering each partition as a vector in $\mathbb{Z}^n$ and taking successive differences. Explicitly, the highest weight word corresponding to $\mathcal{O}$ is*

$$\mu^1 - \mu^0, \mu^2 - \mu^1, \dots, \mu^r - \mu^{r-1}.$$

A now classic bijection due to Sundaram [13] maps an oscillating tableau $\mathcal{O}$ of length r to a pair $\big(\mathcal{M}(\mathcal{O}), \mathcal{M}_T(\mathcal{O})\big)$, consisting of a matching of a subset of $\{1, \dots, r\}$ and a partial standard Young tableau on the complementary subset. We describe this bijection in Section 4.

Theorem 2.3. *Let $\mathcal{O}$ be an n-symplectic oscillating tableau of length r, not necessarily of empty shape. Then $\mathcal{M}(\operatorname{ev}\mathcal{O})$ is the reversal of $\mathcal{M}(\mathcal{O})$ and $\mathcal{M}_T(\operatorname{ev}\mathcal{O})$ is the Schützenberger evacuation of $\mathcal{M}_T(\mathcal{O})$.*

There is a remarkable geometric description of perfect matchings corresponding to n-symplectic oscillating tableaux of empty shape under Sundaram's bijection: visualise a perfect matching by drawing its pairs as (straight) diagonals connecting the vertices of a labelled regular r-gon. Then a perfect matching is $(n+1)$-noncrossing, and the image of an n-symplectic oscillating tableau, if it contains at most n pairs that mutually cross in this picture.

Theorem 2.4. *The bijection $\mathcal{M}$ between n-symplectic oscillating tableaux of empty shape and $(n+1)$-noncrossing perfect matchings intertwines promotion and rotation, and evacuation and reversal:*

$$\operatorname{rot}\mathcal{M}(\mathcal{O}) = \mathcal{M}(\operatorname{pr}\mathcal{O}) \text{ and } \operatorname{rev}\mathcal{M}(\mathcal{O}) = \mathcal{M}(\operatorname{ev}\mathcal{O}).$$

2.2 The adjoint representation of the general linear groups

Definition 2.5 (Stembridge [12])**.** *A* $\mathrm{GL}(n)$-alternating tableau *of length r and weight μ is a sequence of vectors in $\mathbb{Z}^n$*

$$\emptyset = \mu^0, \mu^1, \dots, \mu^{2r} = \mu$$

such that

the entries in each vector are weakly decreasing,

for even i, μ^{i+1} is obtained from μ^i by adding 1 to an entry, and

for odd i, μ^{i+1} is obtained from μ^i by subtracting 1 from an entry.

Proposition 2.6. *Let C be the crystal corresponding to $\otimes^r \mathrm{GL}(n)$, where $\mathrm{GL}(n)$ is the adjoint representation of the general linear group $\mathrm{GL}(n)$. Then the highest weight words of C are obtained from $\mathrm{GL}(n)$-alternating tableaux by taking successive differences. Explicitly, the highest weight word corresponding to $\mathcal{A}$ is the sequence of r pairs*

$$(\mu^1 - \mu^0, \mu^2 - \mu^1), \dots, (\mu^{2r-1} - \mu^{2r-2}, \mu^{2r} - \mu^{2r-1}).$$

It is tempting to regard each vector in an alternating tableau as a pair of partitions by separating the positive and negative terms. Indeed, this is what we will do below. However, for $n > 2$ promotion does not preserve the number of non-zero entries of an alternating tableau. In fact, it is not clear whether there is an embedding ι of the set of $\mathrm{GL}(n)$-alternating tableaux for into the set of $\mathrm{GL}(n+1)$-alternating tableaux such that $\mathrm{pr}\,\iota(\mathcal{A}) = \mathrm{pr}\,\mathcal{A}$.

In Section 4, we introduce a bijection similar in spirit to Sundaram's, that maps an alternating tableau $\mathcal{A}$ of length r to a triple $\big(\mathcal{P}(\mathcal{A}), \mathcal{P}_P(\mathcal{A}), \mathcal{P}_Q(\mathcal{A})\big)$, consisting of a bijection between two subsets, R and S, of $\{1, \dots, r\}$, and two partial standard Young tableaux. The shapes of these tableaux are obtained by separating the positive and negative terms in the weight of the alternating tableaux. The entries of the first tableaux then form the complementary subset of R, the entries of the second form the complementary subset of S.

Theorem 2.7. *Let $\mathcal{A}$ be a $\mathrm{GL}(n)$-alternating tableau of length $r \le \lfloor \frac{n+1}{2} \rfloor$, not necessarily of empty weight. Then $\mathcal{P}(\mathrm{ev}\,\mathcal{A})$ is the reversal of the complement of $\mathcal{P}(\mathcal{A})$ and*

$$\big(\mathcal{P}_P(\mathrm{ev}\,\mathcal{A}), \mathcal{P}_Q(\mathrm{ev}\,\mathcal{A})\big) = \big(\mathrm{ev}\,\mathcal{P}_P(\mathcal{A}), \mathrm{ev}\,\mathcal{P}_Q(\mathcal{A})\big).$$

Theorem 2.8. *For $n \ge r-1$ and also for $n \le 2$ the bijection $\mathcal{P}$ between $\mathrm{GL}(n)$-alternating tableaux of empty weight and permutations intertwines rotation and promotion:*

$$\mathrm{rot}\,\mathcal{P}(\mathcal{A}) = \mathcal{P}(\mathrm{pr}\,\mathcal{A}).$$

For odd $n \ge r$ and for even $n \ge r-1$, it intertwines reverse-complement and evacuation:

$$\mathrm{rc}\,\mathcal{P}(\mathcal{A}) = \mathcal{P}(\mathrm{ev}\,\mathcal{A}).$$

3 The cactus groups and local rules

Definition 3.1. *The r-*fruit cactus group*, $\mathfrak{C}_r$, has generators $s_{p,q}$ for $1 \leq p < q \leq r$ and defining relations*

- $s_{p,q}^2 = 1$
- $s_{p,q}\, s_{k,l} = s_{k,l}\, s_{p,q}$ *if* $q < k$ *or* $l < p$
- $s_{p,q}\, s_{k,l} = s_{p+q-l,p+q-k}\, s_{p,q}$ *if* $p \leq k < l \leq q$

The following lemma shows that it is sufficient to define the action of the composite $s_{1,q}\, s_{2,q}$. The first relation was observed by White [15, lem. 2.3], the second is essentially how Schützenberger [10, sec. 5] introduced evacuation initially.

Lemma 3.2. *We have*

$$s_{p,q} = s_{1,q}\, s_{1,q-p+1}\, s_{1,q} \quad \textit{and} \quad s_{1,q} = s_{1,2}\, s_{2,2}\;\; s_{1,3}\, s_{2,3}\; \ldots\; s_{1,q}\, s_{2,q}\,.$$

Henriques and Kamnitzer [3] defined the action of the cactus group on r-fold tensor products of crystals in terms of Lusztig's involution introduced in [6]. For semistandard Young tableaux this involution is precisely Schützenberger's evacuation. However, we follow Lenart's approach [5] and define the action of the cactus group in terms of van Leeuwen's local rules [4, Rule 4.1.1], which generalise Fomin's [11, A 1.2.7].

Definition 3.3. *Let λ be any weight of a Lie group with Weyl group W. Then $\mathrm{dom}_W(\lambda)$ is the dominant representative of the W-orbit $W\lambda$.*

Let A be a crystal and B and C be crystals of minuscule representations. Then the local rule

$$\tau^A_{B,C} \colon A \otimes B \otimes C \to A \otimes C \otimes B$$

is a weight preserving bijection defined for highest weight words $\alpha \otimes \beta \otimes \gamma$ as follows: let κ be the weight of α, let λ be the weight of $\alpha \otimes \beta$ and let ν be the weight of $\alpha \otimes \beta \otimes \gamma$. Then

$$\tau^A_{B,C}(\alpha \otimes \beta \otimes \gamma) = \alpha \otimes \gamma' \otimes \beta',$$

where, regarding κ, λ, μ and ν as vectors,

$$\mu = \mathrm{dom}_W(\kappa + \nu - \lambda), \quad \gamma' = \mu - \kappa \quad \textit{and} \quad \beta' = \nu - \mu.$$

We represent this by the following diagram:

$$\begin{array}{ccc} \lambda & \xrightarrow{\gamma} & \nu \\ \beta\uparrow & & \uparrow\beta' \\ \kappa & \xrightarrow[\gamma']{} & \mu \end{array}$$

Figure 2

From now on we omit the labels on the edges, because they are determined by the weights.

Any isomorphism between crystals is determined by specifying a bijection between the corresponding highest weight words. Such a bijection can then be extended to the whole crystal by applying the lowering operators.

Definition 3.4. *Let $C_1, \dots, C_r$ be crystals. Then $\mathrm{s}_{1,q}\,\mathrm{s}_{2,q}$ is the map*

$$\tau_{C_1,C_q}^{C_2\otimes\cdots\otimes C_{q-1}} \circ \cdots \circ \tau_{C_1,C_3}^{C_2} \circ \tau_{C_1,C_2}^{I}$$

from $C_1 \otimes \cdots \otimes C_r$ to $C_2 \otimes \cdots \otimes C_q \otimes C_1 \otimes C_{q+1} \otimes \cdots \otimes C_r$.

Here, to improve readability we omitted identity mappings and wrote $\tau^A_{B,C}$ instead of $\tau^A_{B,C} \otimes 1_D$ when applying the map to a tensor product $A \otimes B \otimes C \otimes D$.

A straightforward induction shows that this definition agrees with Henriques and Kamnitzer's. Our definition may appear at first sight less general than theirs, since we require minuscule representations. However, it turns out that any crystal can be embedded in a tensor product of minuscule or so called quasi-minuscule crystals, for which a local rule is also available. In fact, for Cartan types other than E_8, F_4 and G_2, tensor products of minuscule crystals are sufficient.

In particular, promotion and evacuation of $\mathrm{GL}(n)$-alternating tableaux can be accommodated as follows. Regard the adjoint representation as the tensor product $V \otimes V^*$, where V is the vector representation of $\mathrm{GL}(n)$ and V^* is its dual. Both of these are minuscule. Thus, we first apply $\mathrm{s}_{1,2r}\,\mathrm{s}_{2,2r}$ to the word interpreted as an element of $C_1 \otimes \cdots \otimes C_{2r}$ with $C_1 = V, C_2 = V^*, \dots$ Then we apply $\mathrm{s}_{1,2r}\,\mathrm{s}_{2,2r}$ again. We call the result of this operation the promotion of the original word.

To illustrate, let us compute the promotion of a $\mathrm{GL}(3)$-alternating tableau. The Weyl group of $\mathrm{GL}(n)$ is the symmetric group $\mathfrak{S}_n$, so dom_W is just returning its argument sorted in decreasing order. The first row is the original alternating tableau. Regarding the sequence of successive differences of the vectors in the first row as an element of $\bigotimes^5(V \otimes V^*)$, the second row is obtained by applying $\mathrm{s}_{1,10}\,\mathrm{s}_{2,10}$. Repeating this one more time, we obtain the promotion. For better readability we write $\bar{1}$ in place of -1.

$$\begin{array}{cccccccccccccc}
000 & 100 & 10\bar{1} & 20\bar{1} & 2\bar{1}\bar{1} & 20\bar{1} & 2\bar{1}\bar{1} & 20\bar{1} & 10\bar{1} & 100 & 000 & & \\
 & 000 & 00\bar{1} & 10\bar{1} & 1\bar{1}\bar{1} & 10\bar{1} & 1\bar{1}\bar{1} & 10\bar{1} & 00\bar{1} & 000 & 00\bar{1} & 000 & \\
 & & 000 & 100 & 10\bar{1} & 11\bar{1} & 10\bar{1} & 11\bar{1} & 10\bar{1} & 100 & 10\bar{1} & 100 & 000
\end{array} \tag{3.1}$$

The four vectors in the square demonstrate that the naive embedding of $\mathrm{GL}(n)$-alternating tableaux into the set of $\mathrm{GL}(n+1)$-alternating tableaux is not compatible with promotion, as mentioned in Section 2.2: padding the vectors in the square corresponding to κ, λ and

ν in Figure 2 with zeros, and applying the local rule, we obtain the square

$$\begin{matrix} 100\bar{1} & 10\bar{1}\bar{1} \\ 110\bar{1} & 11\bar{1}\bar{1}, \end{matrix}$$

with $\mu = 11\bar{1}\bar{1}$, rather than the vector $100\bar{1}$ one might expect.

4 Growth diagram bijections

In this section we recall Sundaram's bijection (using Roby's description [7]) between oscillating tableaux and matchings. We also present a new bijection, in the same spirit, between alternating tableaux and partial permutations. In both cases, the action of the cactus group on highest weight words becomes particularly transparent when using Fomin's growth diagrams and local rules for the Robinson-Schensted correspondence.

We give a slightly non-standard presentation with the benefit that these local rules can be regarded as a variation of the classical case of Definition 3.3.

$$c = \begin{tikzcd} \lambda \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \nu \arrow[d] \\ \kappa \arrow[r] & \mu \end{tikzcd} \qquad \text{or} \qquad c = \begin{tikzcd} \lambda \arrow[r] \arrow[d] \arrow[dr, phantom, "\boxed{\times}"] & \lambda \arrow[d] \\ \lambda \arrow[r] & \mu \end{tikzcd}$$

$$\mu' = \mathrm{dom}_{\mathfrak{S}_n}(\kappa' + \nu' - \lambda') \qquad\qquad \mu = \lambda + \epsilon_1$$

$$\lambda' = \mathrm{dom}_{\mathfrak{S}_n}(\kappa' + \nu' - \mu') \qquad\qquad \lambda = \mu - \epsilon_1$$

Figure 3: Cells of a growth diagram and corresponding local rules.

In general, a growth diagram is a finite collection of cells (as in Figure 3, where a prime denotes the conjugate partition), arranged in the form of a Ferrers diagram. Thus, for each cell in the diagram all cells above and to the left are also present. The four corners of each cell c are labelled with partitions as indicated.

A difference to Definition 3.3 is that two adjacent partitions (as for example λ and κ in Figure 3) either coincide or the one at the head of the arrow is obtained from the other by adding a single box. In the latter case, we write $\lambda \lessdot \kappa$ and, if κ is obtained by adding one to the i-th part, $\kappa = \lambda + \epsilon_i$.

The two rules in Figure 3 determining μ are called *forward rules*, the two rules in Figure 3 determining λ are called *backward rules*.

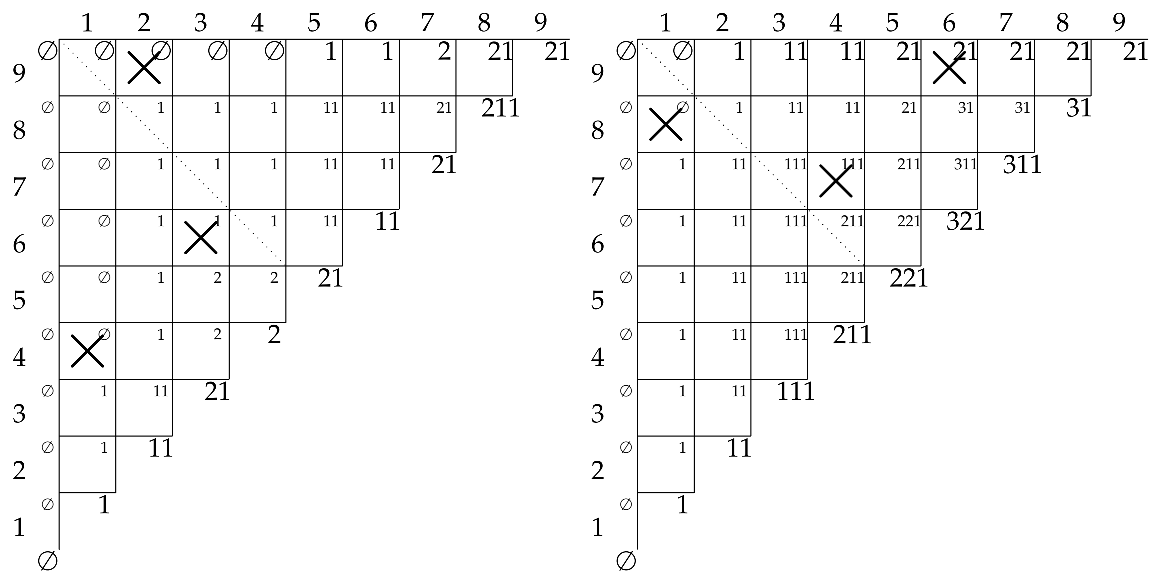

Figure 4: A pair of growth diagrams $\mathcal{G}(\mathcal{O})$ and $\mathcal{G}(\mathrm{s}_{1,9}\,\mathcal{O})$ illustrating Theorem 2.3. The dotted line indicates the axis of reflection for the matchings $\mathcal{M}(\mathcal{O})$ and $\mathcal{M}(\mathrm{s}_{1,9}\,\mathcal{O})$.

4.1 Roby's description of Sundaram's correspondence

Definition 4.1. *Let $\mathcal{O} = (\mu_0, \mu_1, \ldots, \mu_r)$ be an oscillating tableau. The associated triangular growth diagram $\mathcal{G}(\mathcal{O})$ consists of r left-justified rows, with $i-1$ cells in row i for $i \in \{1, \ldots, r\}$, where row 1 is the bottom row. Label the cells according to the following specification:*

- R1 *Label the corners of the cells along the diagonal from south-west to north-east with the partitions in $\mathcal{O}$.*
- R2 *Label the corners of the subdiagonal with the smaller of the two partitions labelling the two adjacent corners on the diagonal.*
- R3 *Use the backward rules to determine which cells contain a cross.*

Let $\mathcal{M}(\mathcal{O})$ be the matching containing a pair $\{i, j\}$ for every cross in column i and row j of the $\mathcal{G}(\mathcal{O})$. Furthermore, let $\mathcal{M}_T(\mathcal{O})$ be the partial standard Young tableau corresponding to the sequence of partitions along the top border of $\mathcal{G}(\mathcal{O})$.

An example for this procedure, which also illustrates Theorem 2.3, can be found in Figure 4. Let w be the highest weight word $1, 2, 1, -2, 2, -1, 1, 3, -3$. The partitions in the corresponding 3-symplectic oscillating tableau $\mathcal{O}$ label the corners of the diagonal of the growth diagram on the left hand side. Applying the backward rules, we obtain the matching and the partial standard Young tableau

$$\mathcal{M}(\mathcal{O}) = \{\{1,4\},\{2,9\},\{3,6\}\} \text{ and } \mathcal{M}_T(\mathcal{O}) = \begin{array}{|c|c|}\hline 5 & 7 \\ \hline 8 & \\ \cline{1-1}\end{array}.$$

Using Lemma 3.2 and Definition 3.4 one can compute that $\mathrm{s}_{1,9}\, w$ is the highest weight word $1,2,3,1,2,1,-2,-3,-1$ corresponding to the 3-symplectic oscillating tableau labelling the corners of the diagonal of the growth diagram on the right hand side. Applying the backward rules again, we obtain the matching and the partial standard Young tableau predicted by Theorem 2.3:

$$\mathcal{M}(\mathrm{s}_{1,9}\,\mathcal{O}) = \{\{1,8\},\{4,7\},\{6,9\}\} \text{ and } \mathcal{M}_T(\mathrm{s}_{1,9}\,\mathcal{O}) = \begin{array}{|c|c|}\hline 2 & 5 \\ \hline 3 & \\ \cline{1-1}\end{array}.$$

4.2 A new variant for Stembridge's alternating tableaux

Definition 4.2. *Let $\mathcal{A} = \mu^0, \mu^1, \dots, \mu^{2r}$ be an alternating tableau. The associated growth diagram $\mathcal{G}(\mathcal{A})$ is an $r \times r$ square of cells, obtained as follows:*

P0 Transform $\mathcal{A}$ into a sequence of pairs of partitions (staircases in Stembridge's terminology)

$$(\pi^0, \nu^0), (\pi^1, \nu^1), \dots, (\pi^{2r}, \nu^{2r}).$$

Each vector μ^i is a weakly decreasing sequence of integers. We obtain the positive part of the staircase, the partition π^i from μ^i by removing all entries less than or equal to zero. The negative part of the staircase, ν^i is obtained from μ^i by removing all entries greater than or equal to zero, reversing the signs of the remaining entries and reversing the sequence.

P1 Label the corners of the cells along the diagonal from north-west to south-east with these staircases.

P2 Apply the backward rules (rotated counterclockwise by 90°) on the positive parts of the staircases to determine which cells below the diagonal contain a cross.

P3 Use the backward rules (rotated clockwise by 90°) on the negative parts of the staircases to determine which cells above the diagonal contain a cross.

Let $\mathcal{P}(\mathcal{A})$ be the partial permutation mapping i to j for every cross in column i and row j of $\mathcal{G}(\mathcal{A})$, and let $(\mathcal{P}_P(\mathcal{A}), \mathcal{P}_Q(\mathcal{A}))$ be the pair of partial standard Young tableau corresponding to the sequence of partitions along the bottom and the right border of $\mathcal{G}(\mathcal{A})$, respectively.

An example for this procedure, which also illustrates Theorem 2.7, can be found in Figure 5. Let w be the $\mathrm{GL}(10)$ highest weight word

$$(e_1, -e_{10}), (e_1, -e_{10}), (e_{10}, -e_9), (e_9, -e_{10}), (e_{10}, -e_{10}), (e_2, -e_{10}),$$

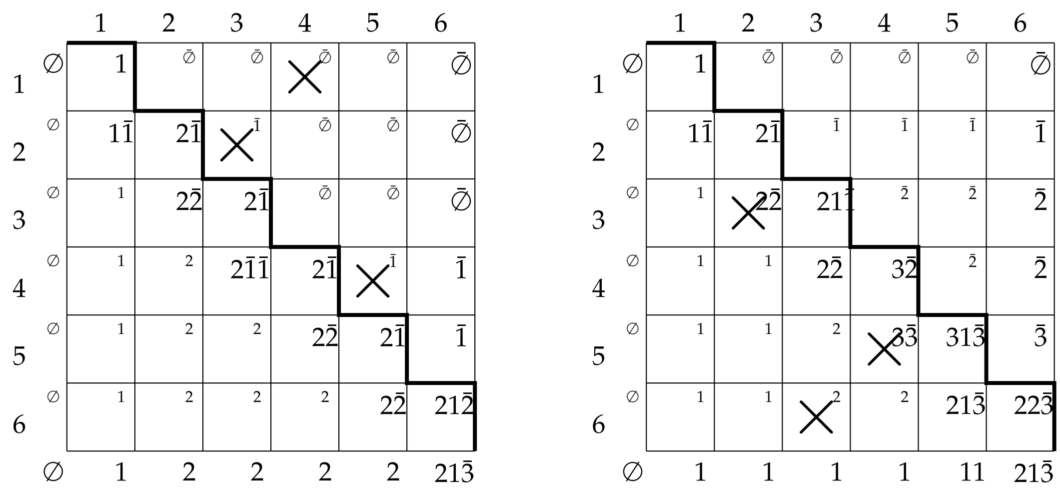

Figure 5: A pair of growth diagrams $\mathcal{G}(\mathcal{A})$ and $\mathcal{G}(\mathrm{s}_{1,6}\,\mathcal{A})$ illustrating Theorem 2.7.

where e_i is the i-th standard vector. The staircases in the corresponding alternating tableau $\mathcal{A}$ label the corners of the diagonal of the growth diagram on the left hand side, where we write the negative partitions with bars. Applying the backward rules, we obtain the partial permutation and the partial standard Young tableaux

$$\mathcal{P}(\mathcal{A}) = \{(3,2),(4,1),(5,4)\}, \mathcal{P}_P(\mathcal{A}) = \boxed{3\,|\,5\,|\,6}, \text{ and } \mathcal{P}_Q(\mathcal{A}) = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 6 \\ \cline{1-1}\end{array}.$$

On the right hand side of the figure the growth diagram obtained by applying the same procedure to $\mathrm{s}_{1,6}\,w$, which yields

$$\mathcal{P}(\mathcal{A}) = \{(2,3),(3,6),(4,5)\}, \mathcal{P}_P(\mathcal{A}) = \boxed{1\,|\,2\,|\,4}, \text{ and } \mathcal{P}_Q(\mathcal{A}) = \begin{array}{|c|c|}\hline 1 & 6 \\ \hline 5 \\ \cline{1-1}\end{array}.$$

as predicted by Theorem 2.7.

Finally, consider the $\mathrm{GL}(3)$-alternating tableau in the first row of (3.1), which corresponds to the permutation depicted in Figure 1. However, its promotion, as computed in the last row of (3.1), is different from the alternating tableau corresponding to the rotated permutation. Indeed, the hypothesis of Theorem 2.8 is not satisfied.

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